selected, the probability in each succeeding trial of successfully picking a $(k+1)$st unused number is $p = (n-k)/n$. We are interested in the total number of trials before all numbers from 1 to n have been used. ∎

EXAMPLE 7 The Poisson Distribution

A large computer receives, on average, λ new jobs a minute. What is the probability that exactly k new jobs will come in during a given minute?

Solution There are many problems of this sort. Another is: If the telephone company handles, on average, λ calls a minute, what is the probability that, in the course of a day, at some point they will need k circuits (if no caller is to receive an "all circuits busy" signal)?

Problems of this sort are usually modeled very well by the following probability distribution:

$$f_\lambda(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k = 0, 1, 2, \ldots. \tag{8}$$

The distribution with this function is called the **Poisson distribution**. Sorry about the λ (a Greek lambda); a roman letter would do fine (so long as it does not make you think the value must be an integer, or make you think that it varies), but λ is traditional. Also in Eq. (8), e is the famous base of the natural logarithms, $2.71828\ldots$.

Where does Eq. (8) come from? It is the limit of a Bernoulli trials model. To see this, assume that at most one new computer job comes in per second. If, on average, λ new jobs come in per minute, that suggests that the probability of receiving a new job in any given second is $\lambda/60$. From this we can calculate the probability that new jobs begin during exactly k of the 60 seconds — it's just the probability of k successes (job starts) in 60 Bernoulli trials (the seconds).

The reason this Bernoulli model isn't good enough is that it doesn't allow more than one new job to come in each second. So, let's divide seconds into tenths, and model the situation again with 600 segments in a minute instead of 60. If we keep increasing the number of segments, in the limit there is no bound on the number of jobs that could come in per minute (or per any nonzero time period). The formula we get in the limit is Eq. (8).

To show that we get Eq. (8), we need a fact from calculus about e and some familiarity with limit calculations. The fact from calculus is:

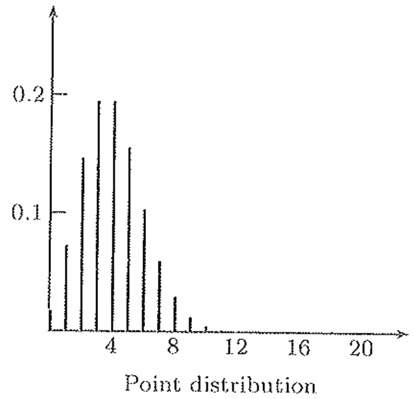
$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n.$$

We use it as follows. Fix λ and let n be the number of intervals into which the single Poisson period is to be divided in a Bernoulli-trials approximation. Then the probability of "success" in each interval is $p = \lambda/n$. Hence, the probability of k successes in the n intervals in the binomial approximation is

$$\begin{aligned}\binom{n}{k}p^k q^{n-k} &= \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}\\ &= \binom{n}{k}\left(\frac{\lambda/n}{1-\frac{\lambda}{n}}\right)^k\left(1-\frac{\lambda}{n}\right)^n.\end{aligned}$$

Now we factor out the terms independent of n, namely $\lambda^k/k!$, let n go to infinity, use the fact above from calculus, and simplify – in the limit many of the quantities in the first two factors cancel [27]. (In the last part of this section and then later in Section 6.7, we consider another way of looking at the binomial distribution as the number of trials n goes to infinity.)

Fig. 6.9 shows the Poisson point distribution for $\lambda = 4$. If we were to approximate this by a binomial distribution with $n = 20$, then p would be $4/20 = .2$. So the approximation is $B_{20,.2}$, exactly the binomial distribution in Fig. 6.7. Compare! ∎



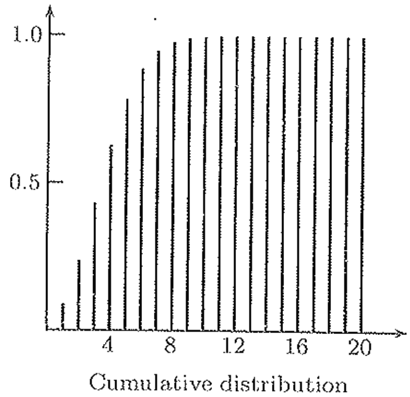


FIGURE 6.9

The Poisson probability distribution for $\lambda = 4$ (k shown up to 20).